A Stable Direct Solution of Perspective-Three-Point Problem

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The perspective-three-point problem (P3P) is a classical problem in computer vision. The existing direct solutions of P3P have at least 3 limitations: (1) the numerical instability when using different vertex permutations, (2) the degeneration in the geometric singularity case, and (3) the dependence on particular equation solvers. A new direct solution of P3P is presented to deal with these limitations. The main idea is to reduce the number of unknown parameter by using a geometric constraint we called “perspective similar triangle” (PST). The PST method achieves high stability in the permutation problem and in presence of image noise, and does not rely on particular equation solvers. Furthermore, reliable results can be retrieved even in “danger cylinder”, a typical kind of geometric singularity of P3P, where all existing direct solutions degenerate significantly.

Keywords: Perspective-three-point problem (P3P); camera pose estimation; geometric singularity.

1. INTRODUCTION

The term “perspective-three-point problem” (P3P) is coined by Fischler and Bolles\textsuperscript{1} for the problem of determining the pose of a calibrated camera from 3 correspondences between 3D target points and their 2D projections. The P3P problem has a long history since 1841\textsuperscript{2} and has many applications in computer vision\textsuperscript{3,4,5}, photogrammetry\textsuperscript{6,7}, robotics\textsuperscript{8,9}, etc.

P3P attracts many researchers, because it is the smallest subset of perspective-n-point problem (PnP)\textsuperscript{5,4} and yields multiple solutions\textsuperscript{10,11}. Haralick et al.\textsuperscript{3} reviewed many old and new variants of the basic three-point methods and carefully examined their numerical stability. Gao et al.\textsuperscript{10} obtained a complete solution classification using Wu-Ritt’s zero decomposition method. DeMenthon and Davis\textsuperscript{12} showed that the camera pose can be estimated efficiently by using approximations of the perspective. Fischler and Bolles\textsuperscript{1} presented the RANSAC algorithm, which divided the n-point

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problem into 3-point subsets and eliminated the multiplicity by checking consistency. Quan and Lan\cite{13} reviewed the 3-point algebraic method and developed linear algorithms to handle the problems with more than 4 points. Triggs, Ameller, and Quan\cite{14} surveyed several existing methods for pose estimation, and then introduced four new linear algorithms for 3-point and 4-point problems by SVD null space estimation. Zhi and Tang\cite{15} presented a stable linear algorithm for 4-point problem which is easy to implement. Abidi and Chandra\cite{8} proposed a solution from 4 coplanar points for a perspective camera with unknown focal length. Bujnak, Kukelova, and Pajdla\cite{16} generalized the solution of Abidi et.al. to 4 arbitrary points. Ansar and Daniilidis\cite{17} presented linear solutions for both n points and n lines. Lepetit, Moreno-Noguer and Fua\cite{4} proposed a O(n) non-iterative algorithm which considered the solution as weighted sum of null eigenvectors and achieved excellent results for the n \geq 5 case.

The solutions of P3P can be classified into direct solution, iterative solution, geometric method, complete solution classification, etc. Comparing with other methods, the direct solution is widely used for camera pose estimation in practice because it performs efficiently and can be easily implemented. The direct solution of P3P has been well developed since 1841\cite{2,18,1,19,20,3}, however, to the best of our know, the existing direct solutions have at least 3 limitations:

(1) The permutation problem. The great work of Haralick et.al.\cite{3} showed that the permutation of triangle vertex significantly affected the numerical stability of the solution. For the same input data, the accuracy of result varies about 10^3 times depending on the chosen permutation.

(2) The geometric singularity problem. Geometric singularity\cite{3} is caused by some unstable geometric structure, in which a small change in the position of perspective center will lead to a big change in the result. In the case of “danger cylinder”, a typical kind of geometric singularity, the results of all existing direct solutions become instable significantly due to the large estimation error.

(3) The equation solver problem. The essence of the direct solution of P3P is to convert the problem into a quartic equation. Different methods lead to different quartic equations. We find that if the equation solver chosen is not appropriate, for example, the widely used analytical solution of quartic equation, the performances of the existing direct solutions degenerate significantly. It is useful to develop a direct solution which is numerical stable without relying on specific equation solvers, because choosing and implementing an appropriate equation solver would increase the burden on application developers.

The reason of the limitations of existing direct solutions mentioned above may lie in the classic P3P geometric constraint. The overwhelming majority of the direct solutions derive from the classic P3P equation system using the three side lengths of the target triangle as geometrical constraint, namely, the distances among the three estimated control points are constrained to equal the side lengths of the tar-
get triangle. DeMenthon et al.\textsuperscript{12} developed a new solution which did not derive from the classic P3P geometrical constraint and provided a uniform framework for different perspective approximations. The efficiency of their solution was significantly accelerated by approximation, but the accuracy was not enhanced due to the approximation. For the P4P problem, the great work of Bujnak, Kukelova, and Pajdla\textsuperscript{16} used the ratios of the side lengths as constraints to reduce the number of unknown parameters, and achieved a general solution for P4P with unknown focal length. The numerical performance of their solution was greatly enhanced by using Grobner basis technique as equation solver.

The paper presented a new direct solution of P3P problem with high numerical stability and accuracy. The advantages of our direct solution are as follows: (1) it is numerical stable for the permutation problem. (2) Reliable results can be retrieved in the geometric singularity case. (3) It is robust for different equation solvers, and (4) is stable in presence of image noise.

The rest of the paper is organized as follows: the procedure of our solution is described in section 2, the initial equation system of P3P and that of PST are compared in section 3, and the experimental results are provided in section 4.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{The geometry of the P3P problem.}
\end{figure}

### 2. Our Approach to the P3P Problem

#### 2.1. Converting P3P into PST

The geometry of the P3P problem is shown in Figure 1. The 3D control points $p_0$, $p_1$, $p_2$ of target triangle is projected to the image plane as $m_0$, $m_1$, $m_2$. The triangle side lengths $D_1 = |p_0p_1|$, $D_2 = |p_0p_2|$, $D_3 = |p_1p_2|$ and the coordinates
of the projected image point \( m \) are known. The purpose is to solve the unknown depth \( d_i \) (\( i = 0, 1, 2 \)) of \( p_i \) to the perspective center \( O \).

We convert the P3P into an equivalent problem we called “perspective similar triangle” (PST). Given a base point \( p'_0 \) lies on the line \( Om_0 \), the purpose of PST is to determine the unknown depths \( d'_i \) of the vertexes \( p'_1 \) and \( p'_2 \), which make the triangle \( p'_0 p'_1 p'_2 \) be similar to the target \( p_0 p_1 p_2 \).

It is easy to prove that, for any triangular target \( p_0 p_1 p_2 \) projected in image plane as \( m_0 m_1 m_2 \), there exists a unique similar triangle \( p'_0 p'_1 p'_2 \) which pass the base point \( p'_0 \) and be with the same image projection. By constructing a plan passing \( p'_0 \) and being parallel to \( p_0 p_1 p_2 \), the vertexes \( p'_1 \) and \( p'_2 \) can be determined by the intersections of the parallel plan and the lines \( Om_1, Om_2 \). Therefore, the number of solutions of PST is the same as that of the original P3P problem.

When the depths \( d'_i \) of \( p'_0 p'_1 p'_2 \) are solved, the depths \( d_i \) of \( p_0 p_1 p_2 \) can be retrieved easily by multiplying a scale factor on \( d'_i \),

\[
d_i = \lambda d'_i
\]

in which \( \lambda \) denotes the scale factor of the similar triangles.

![Fig. 2. The geometry of an equivalent problem “perspective similar triangle” (PST).](image-url)
2.2. Solution of PST

The geometry of PST is shown in Figure 2. \(H_1\) and \(H_2\) are two points on the lines \(O\|_1, O\|_2\) respectively, and \(p'_0H_1 \perp OH_1, p'_0H_2 \perp OH_2\). \(l_0\) denotes the distance from the perspective center \(O\) to the base point \(p'_0\), \(l_1\) denotes the distance \(|OH_1|\), and \(l_2\) denotes the distance \(|OH_2|\). Without lose of general, we set the depth of the base point \(p'_0\) as \(l_0 = |OP'_0| = 1\). Let \(t_1\) denotes the distance from \(H_1\) to \(p'_1\), and \(t_2\) denotes the distance from \(H_2\) to \(p'_2\). The depths \(d'_i\) \((i = 0, 1, 2)\) of \(p'_i\) can be expressed by \(t_1\) and \(t_2\) as

\[
d'_0 = l_0, \quad d'_1 = l_1 + t_1, \quad d'_2 = l_2 + t_2
\]

with

\[
l_0 = 1, \quad l_1 = l_0 \cos \gamma_1, \quad l_2 = l_0 \cos \gamma_2
\]

And then, the three unknown variables of P3P are reduced to two, \(t_1\) and \(t_2\).

As \(p'_0H_1 \perp OH_1\) and \(p'_0H_2 \perp OH_2\), the side lengths of \(p'_0p'_1p'_2\) are

\[
|p'_0p'_1| = \sqrt{t_1^2 + C_1^2}, \quad |p'_0p'_2| = \sqrt{t_2^2 + C_2^2}
\]

with

\[
C_1 = l_0 \sin \gamma_1, \quad C_2 = l_0 \sin \gamma_2
\]

in which, \(C_1\) denotes the length of \(|p'_0H_1|\), and \(C_2\) the length of \(|p'_0H_2|\).

Because \(p'_0p'_1p'_2\) is similar to \(p_0p_1p_2\), two geometrical constraints can be used to form the initial equation system:

**A.** The first constraint is the ratios of the corresponding side lengths of similar triangles, \(|p'_0p'_1|/|p_0p_1| = |p'_0p'_2|/|p_0p_2|\). By substituting (3) into the equation, squaring on both sides, and moving the terms from the right to the left, we obtain

\[
A_1t_1^2 - t_2^2 + A_2 = 0
\]

with

\[
k = D_2/D_1, \quad A_1 = k^2, \quad A_2 = k^2C_1^2 - C_2^2
\]

**B.** The second constraint is that the corresponding angles of similar triangles equal each other, namely, \(\cos \angle p'_1p'_0p'_2 = \cos \angle p_1p_0p_2\). According to the law of cosine,

\[
\cos \angle p_1p_0p_2 = \frac{D_1^2 + D_2^2 - D_3^2}{2D_1D_2}
\]

\[
\cos \angle p'_1p'_0p'_2 = \frac{\overrightarrow{p'_0p'_1} \cdot \overrightarrow{p'_0p'_2}}{|\overrightarrow{p'_0p'_1}| |\overrightarrow{p'_0p'_2}|}
\]

Then, we can get that,

\[
\overrightarrow{p'_0p'_1} \cdot \overrightarrow{p'_0p'_2} = |\overrightarrow{p'_0p'_1}| |\overrightarrow{p'_0p'_2}| \frac{D_1^2 + D_2^2 - D_3^2}{2D_1D_2}
\]
with
\[ \begin{align*}
\vec{p}_0\vec{p}_1 &= \vec{OP}_1 - \vec{OP}_0 = (t_1 + t_4)\vec{v}_0^1 - t_0\vec{v}_0^0 \\
\vec{p}_0\vec{p}_2 &= \vec{OP}_2 - \vec{OP}_0 = (t_2 + t_3)\vec{v}_0^2 - t_0\vec{v}_0^0 \\
|\vec{p}_0\vec{p}_1| &= |\vec{p}_0\vec{p}_2| = |k|\vec{p}_0\vec{p}_1|^2 = k(t_1^2 + C_1^2) 
\end{align*} \]

in which \( \vec{v}_0^i \) denotes the unit vector on \( \vec{OP}_i \) \((i = 0, 1, 2)\) direction. “.” denotes the dot product operation, \( \vec{v}_0^1 \cdot \vec{v}_0^1 = \cos \gamma_1, \vec{v}_0^3 \cdot \vec{v}_0^3 = \cos \gamma_2, \vec{v}_0^4 \cdot \vec{v}_0^4 = \cos \gamma_3, \) and \( \vec{v}_0^1 \cdot \vec{v}_0^3 = 0. \)

By substituting (6) into (4), the unknown variable \( t \) with \( \vec{v} \) is obtained, and a fourth order polynomial of \( t \) is obtained,
\[ \begin{align*}
A_3t_1 + A_4t_2 + A_5t_1t_2 + A_6t_1^2 + A_7 &= 0 
\end{align*} \]

where
\[ \begin{align*}
A_3 &= t_2 \cos \gamma_3 - t_1 \\
A_4 &= t_1 \cos \gamma_3 - t_2 \\
A_5 &= \cos \gamma_3 \\
A_6 &= (D_3^2 - D_2^2 - D_1^2)/(2D_1^2) \\
A_7 &= t_0^2 - t_1^2 - t_2^2 + t_1t_2 \cos \gamma_3 + A_6C_1^2
\end{align*} \]

By combining (4) and (7), we obtain the initial equation system of PST,
\[ \begin{align*}
\begin{cases}
A_1t_1^2 - t_2^2 + A_2 = 0 \\
A_3t_1 + A_4t_2 + A_5t_1t_2 + A_6t_1^2 + A_7 = 0
\end{cases}
\end{align*} \]

We can derive from (7) that,
\[ t_2 = -\frac{A_3t_1 + A_4t_1^2 + A_7}{A_4 + A_5t_1} \]

By substituting (9) into (4), the unknown variable \( t_2 \) is eliminated, and a fourth order polynomial of \( t_1 \) is obtained,
\[ \begin{align*}
B_4t_1^4 + B_3t_1^3 + B_2t_1^2 + B_1t_1 + B_0 &= 0 
\end{align*} \]

with
\[ \begin{align*}
B_4 &= A_3^2 - A_1A_5^2 \\
B_3 &= 2(A_3A_6 - A_1A_4A_5) \\
B_2 &= A_1^2 + 2A_6A_7 - A_1A_5^2 - A_2A_5^2 \\
B_1 &= 2(A_3A_7 - A_2A_4A_5) \\
B_0 &= A_4^2 - A_2A_5^2
\end{align*} \]

\( t_1 \) can be solved from (10), which has at most four solutions. The next step is to determine \( t_2 \) by \( t_1 \). If \( A_4 + A_5t_1 \neq 0 \), then according to (9),
\[ \begin{align*}
t_2 &= -\frac{A_3t_1 + A_4t_1^2 + A_7}{A_4 + A_5t_1}
\end{align*} \]

or if \( A_4 + A_5t_1 = 0 \), then
\[ \begin{align*}
t_2 &= \pm \sqrt{A_1t_1^2 + A_2}
\end{align*} \]
The scale factor of similar triangles can be easily calculated as
\[ \lambda = \frac{|p_0 p_1|}{|p_0' p_1'|} = D_1 / \sqrt{t_1^2 + C_1^2} \] (11)

and then, the depths of the target vertexes \( p_0, p_1, p_2 \) can be determined by (1),
\[ d_0 = \lambda l_0, \quad d_1 = \lambda (l_1 + t_1), \quad d_2 = \lambda (l_2 + t_2) \] (12)

2.3. Multiple Solutions

The camera pose is not uniquely determined from 3-point set due to multiple solutions. The number of the solutions is directly corresponding to the number of real roots of the 4-th order polynomial. To retrieve a unique solution, at least one more point should be introduced. One of the most widely used methods to determine the correct camera pose by using P3P is the RANSAC algorithm, which divide the n-point set into 3-point subsets and check the consistency of these subsets.

It is worth mentioning that the number of solutions of P3P would be inconsistent with the ground truth due to the noise in practice. Let \( F_{true} \) denote an unknown 4-th order polynomial corresponding to the ground truth and \( F_{measured} \) denote the polynomial derived form the measurement. As can be seen in Fig. 3, the plot of \( F_{measured} \) would be disturbed due to the measurement error, and the purpose of P3P is to estimate the roots of the unknown \( F_{true} \) using that of \( F_{measured} \). Assuming \( F_{true} \) has 3 real roots and the root \( x_3 \) being tangent with the x-axis is the true one, \( x_3 \) would probably vanish when noise is introduced. The result is that \( F_{measured} \) returns only 2 solutions and the most important root corresponding to the true pose is missed.

The missing solution problem is determined by the native structure of P3P, and other P3P methods, such as iterative solution, geometric solution and classification methods, also have the same problem.
To compensate the missing solution problem, we estimate the possible missing roots through a simple scheme by using the extrema of $F_{\text{measured}}$. Firstly, we determine the extrema $x_i$ of $F_{\text{measured}}$ by finding the roots of its derivation, a 3-th order polynomial. There exist at most 3 candidates. For each candidate, if $x_i$ is a minimum and $F_{\text{measured}}(x_i) > 0$, or if $x_i$ is a maximum and $F_{\text{measured}}(x_i) < 0$, then $x_i$ is probably to be the missing root. Finally, the probability of $x_i$ being one of the solutions of P3P is checked by its re-projection error on image plane. The smaller the re-projection error corresponding to $x_i$ is, the more possibly the extremum be the missing solution. If the re-projection error of $x_i$ is less than a given threshold, the candidate can be output as additional solutions of P3P.

3. The P3P/PST Equation System

According to the classic P3P constraint, we obtain the classic P3P equation system with three unknown variables,

\[
\begin{align*}
d_1^2 + d_0^2 - 2d_1d_0 \cos \gamma_1 &= D_1^2 \\
d_2^2 + d_0^2 - 2d_2d_0 \cos \gamma_2 &= D_2^2 \\
d_1^2 + d_2^2 - 2d_1d_2 \cos \gamma_3 &= D_3^2
\end{align*}
\] (13)

The classic P3P equation system can be converted in the following form\textsuperscript{10} with two unknown parameters $u$ and $v$,

\[
\begin{align*}
F_1u^2 + v^2 + F_2uv + F_3u + F_4 &= 0 \\
u^2 + F_5v^2 + F_6uv + F_7v + F_8 &= 0
\end{align*}
\] (14)

in which, $F_i (i = 1 \ldots 8)$ denotes the coefficients of the equations. The coefficient $F_i$ varies in different solutions according to the way changing variables.

By converting P3P into PST, only two geometrical constraints are required to form the PST equation system,

\[
\begin{align*}
A_1t_1^2 - t_2^2 + A_2 &= 0 \\
A_3t_1 + A_4t_2 + A_5t_1t_2 + A_6t_1^2 + A_7 &= 0
\end{align*}
\]

The PST equation system is with less complexity than the classic P3P equation system, because it has only two unknown variables. Although (13) can be algebraically simplified into the form as (14), the PST equation system is still less complex, because the first line of the PST equations contains only 3 terms which are $t_1^2$, $t_2^2$ and constant term. It is more convenient to calculate, or to analyze the P3P problem by using PST equations which would lead to better numerical performance. The experimental results will be provided in the next section.

It is worth mentioning DeMenthon and Davis’s novel work\textsuperscript{12} which does not derive its initial equation system from the classic P3P constraint as other direct solutions do. Two side lengths and an angle are used to form initial equations with two unknown variables $t_1$ and $t_2$.

\[
\begin{align*}
\sin t_1/\sin t_2 &= K \\
p(c_1 \sin t_1 + s_1 \cos t_1)(c_2 \sin t_2 + s_2 \cos t_2) + (c_1 \cos t_1 - s_1 \sin t_1)(c_2 \cos t_2 - s_2 \sin t_2) &= q
\end{align*}
\]
To avoid the trigonometric functions $\sin$ and $\cos$ in the equations, the method of DeMenthon and Davis can be seen equivalently as following steps: (1) expanding the variables from two to four ($x_1 = \sin t_1, x_2 = \sin t_2, y_1 = \cos t_1, y_2 = \cos t_2$), and (2) introducing two hidden quadric constraints $x_1^2 + y_1^2 = 1$ and $x_2^2 + y_2^2 = 1$. Therefore, the initial equation system can be expressed equivalently as

$$
\begin{aligned}
&x_1 = K x_2 \\
&a_1x_1x_2 + a_2y_1y_2 + a_3x_1y_2 + a_4x_2y_1 = a_5 \\
&x_1^2 + y_1^2 = 1 \\
&x_2^2 + y_2^2 = 1
\end{aligned}
$$

By isolating $y_2$ on one side of the second equation and squaring both sides, $x_2$ and $y_2$ can be eliminated by using the first and the last line of the equations. This yields an equation system with two unknown variables $x_1, y_1$.

$$
\begin{aligned}
b_1x_1^4 + b_2x_1^3y_1 + b_3x_1^2y_1 + b_4x_1y_1 + b_5 &= 0 \\
x_1^2 + y_1^2 &= 1
\end{aligned}
$$

Comparing with (14), a 4-th order polynomial is involved in (15), while (14) contains only quadric equations. Due to the higher complexity, (15) is not as accurate as the classic P3P equation system.

4. Experimental Results

We compare the accuracy and stability of our approach with that of the existing direct solutions. Following algorithms were compared in our experiments. They are the best algorithms according to the comprehensive tests on P3P problem of Haralick et.al.\textsuperscript{3}.

(A) FB, the method of Fischler and Bolles\textsuperscript{1}. FB is one of the most accurate direct solution of P3P with no algebraic singularity.

(B) LHD, the method of Linnainmaa, Harwood and Davis\textsuperscript{19} with no algebraic singularity.

(C) FS, the method of Finsterwalder and Scheufler\textsuperscript{18}. It is the most accurate direct solution of P3P as summarized in Ref.\textsuperscript{3}, but its drawback is the algebraic singularity.

(D) PST, our solution with no algebraic singularity.

The experimental settings in this paper will follow the comprehensive tests on P3P presented in the great work of Haralick et.al.\textsuperscript{3} Assuming a virtual projective camera with resolution $1024 \times 1024$ and effective focal length $f = 1200$ pixels, the 3D control points $p_0, p_1, p_2$ are projected onto the image plane as $m_0m_1m_2$. The image coordinates of $m_0m_1m_2$ and the side lengths $D_1 = |p_0p_1|$, $D_2 = |p_0p_2|$, $D_3 = |p_1p_2|$ are used as input for each method. The 3D control points $p_0p_1p_2$ are uniformly distributed in the $x, y, z$ interval $[-25,25] \times [-25,25] \times [-24,z_0+24]$, where the parameter $z_0$ denotes the depth of the target. The value of $z_0$ varies from 25 to 125 with interval 10. When $z_0 = 25$, the maximal view angle of the triangles is 90
degrees; when $z_0 = 125$, the maximal view angle is 22.6 degrees. For each value of $z_0$, 5000 random tests are generated.

The calculations were done in double precision. The true depth of the control point $p_i$ was $d^*_i$. We computed the absolute depth error (ADE) of each solution as $E_d = |d^*_0 - d_0| + |d^*_1 - d_1| + |d^*_2 - d_2|$.

4.1. Numerical Accuracy and Stability

The mean absolute depth error (MADE) of each algorithm is shown in Fig. 4, and the standard deviation of ADE is shown in Tables 1. We can see that, our method “PST” performs much more accurately than others in all depth $z_0$.

As pointed out by Haralick\textsuperscript{3}, permutation is one of the most important factors that affects the numerical stability of P3P solutions. For the same test data, the accuracy varies from 1000 to 1 depending on the order of control points used. It is not caused by geometric singularity or other unstable factors, but the algebraic calculations of the solutions. By trying the all 6 possible permutations \{123, 132, 213, 231, 312, 321\}, and comparing the results with the true depth $d^*$, we can find the best permutation and the worst permutation.

In practice it is hard to determine which permutation is the best, because the true depth $d^*$ is unknown. Therefore, it is important to find a solution whose accuracy varies little for different permutations. The mean absolute depth error of the best permutation (MADE\textsubscript{best}) and that of the worst (MADE\textsubscript{worst}) are shown in Fig.5. As can be seen in Fig.5, MADE\textsubscript{best} of our solution is obviously better than other methods. What is more, the MADE\textsubscript{worst} of ours is in the same level of accuracy as the MADE\textsubscript{best} of others.

![Fig. 4. We evaluated the accuracy of the compared methods by plotting the mean absolute depth error (MADE) as a function of depth $z_0$.](image-url)
Figure 5. The MADEs of the compared methods according to the worst permutation and the best permutation.

Table 1. The standard deviation of absolute depth error (ADE).

<table>
<thead>
<tr>
<th>Depth $z_0$</th>
<th>FB</th>
<th>LHD</th>
<th>FS</th>
<th>PST</th>
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<td>115</td>
<td>1.76E-06</td>
<td>2.84E-06</td>
<td>3.34E-07</td>
<td>2.85E-10</td>
</tr>
<tr>
<td>125</td>
<td>2.08E-06</td>
<td>1.91E-06</td>
<td>2.44E-08</td>
<td>3.94E-10</td>
</tr>
</tbody>
</table>

Table 2. The mean and standard deviation of the ADEs of the compared methods in case of “danger cylinder”. “inf” means no result can be retrieved.

<table>
<thead>
<tr>
<th></th>
<th>FB</th>
<th>LHD</th>
<th>FS</th>
<th>PST</th>
</tr>
</thead>
<tbody>
<tr>
<td>MADE</td>
<td>0.04</td>
<td>0.74</td>
<td>inf</td>
<td>2.68E-08</td>
</tr>
<tr>
<td>Std.</td>
<td>1.45</td>
<td>5.26</td>
<td>inf</td>
<td>1.37E-06</td>
</tr>
</tbody>
</table>

4.2. Geometric Singularity

It is a challenge to test the performance of P3P solution in geometric singularity case. The “danger cylinder” is a typical kind of geometric singularity, in which the three control points and the perspective center are on the surface of a circle.
cylinder, and the target triangle is parallel to the image plane.

In this section, the performance of P3P solutions are tested in “danger cylinder”. The experiments are setup as follows: Randomly generate a cylinder \((x-r)^2 + y^2 = r^2\) whose radius parameter \(r\) is uniformly distributed from 5 to 25. The central axis of the cylinder is perpendicular to the image plane, and the perspective center \(O\) is on the surface of the cylinder. Randomly pick 3 control point on the cylinder, and the \(z\) coordinate of the points is uniformly distributed from 25 to 75. For each algorithm, 5000 random tests were performed.

The MADE and the standard deviation in case of “danger cylinder” are shown in Table 2. The degenerations were found in all algorithms. As FS is with algebraic singularity, no result can be retrieved from FS in this case. The MADEs of FB and LHD are 0.04 and 0.74 respectively, and the standard deviations of them are 1.45 and 5.26 respectively. However, The MADE of PST is \(2.68 \times 10^{-8}\), and its result is accurate enough to be seen as reliable.

![Histogram of Eigenvalue Method](image1)

![Histogram of Analytical Solver](image2)

Fig. 6. Histogram of the ADEs of the compared methods when using different equation solvers.

### 4.3. Different Quartic Equation Solvers

As high-order polynomial is involved in the direct solution of P3P, it is important to study the performance of the direct solutions when using different equation solvers. The experimental results in previous sections are based on the eigenvalue method. In this section, another widely used quartic equation solver, the analytical solution of quartic equation, will be tested.

Quartic equation solver was involved in FB, LHD and PST. 10000 random tests
were performed for each solver with the depth parameter $z_0 = 50$. The comparisons between the two different solvers were shown in Fig. 6. In the top of the figure was the histograms of the ADEs of FB, LHD and PST using eigenvalue method, and in the bottom was that of the analytical solver. The accuracy of FB and LHD degenerated significantly when using the analytical solver, but PST worked well with both eigenvalue method and the analytical solver.

### 4.4. Image Noise

In presence of noise, the MADEs and the Std. of the ADEs of the compared methods were tested in both ordinary case and geometric singularity case. 5000 random tests were performed for each method at each noise-level. The $z$ coordinates of the control points were uniformly distributed from 25 to 75. In ordinary case, as shown in Fig. 7, the performance of PST in presence of noise was better than that of other methods. When the tests were performed in the geometric singularity case, as shown in Fig. 8, the advantage of PST against noise became more evidently.
5. Conclusion

The direct solution of P3P has a long history since 1841 but it still remains some important limitations, which are permutation problem, singularity problem, and the dependence on particular equation solvers. Our work can be seen as an useful supplement to this classic subject because the above limitations are overcome by using PST geometrical constraint. The PST method is stable in the permutation problem, and it does not rely on particular equation solvers. Robust results can be retrieved by PST in presence of noise. To the best of our know, there is no existing direct solution can retrieve stable result in geometric singularity case until our approach came out. The experimental results show that reliable results can be retrieved in “danger cylinder” which is a typical kind of geometric singularity.

References

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